Poisson structures on the Teichmüller space of hyperbolic surfaces with conical points

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Abstract. In this paper two Poisson structures on the moduli space of hyperbolic surfaces with conical points are compared: the Weil-Petersson one and the η coming from the representation variety. We show that they are multiple of each other, if the angles do not exceed 2π. Moreover, we exhibit an explicit formula for η in terms of hyperbolic lengths of a suitable system of arcs.

1. Introduction

The uniformization theorem for hyperbolic surfaces of genus g with conical points ([**McO88**], [**McO93**] and [**Tro91**]; see Section 2) allows to identify the space $\mathcal{Y}(S,x)(\vartheta)$ of hyperbolic metrics on S (up to isotopy) with angles $\vartheta = (\vartheta_1, \dots, \vartheta_n)$ at the marked points $x = (x_1, \dots, x_n)$ to the Teichmüller space $\mathcal{T}(S,x)$ (see Section 3).

It is thus possible to define a Weil-Petersson pairing $h^*_{WP,\vartheta} = g^*_{WP,\vartheta} + i\eta_{WP,\vartheta}$ on the cotangent space of $\mathcal{T}(S,x)$ at J as

$$h^*_{WP,\vartheta}(\varphi,\psi) := -\frac{1}{4}\int_S g_\vartheta^{-1}(\varphi,\overline{\psi})$$

where $\varphi, \psi \in H^0(S, K_S^{\otimes 2}(x)) \cong T^*\mathcal{T}(S,x)$ are holomorphic with respect to J and g_ϑ is the area form of the unique hyperbolic metric conformally equivalent to J and with angles ϑ. In particular, $h^*_{WP,0}$ is the standard Weil-Petersson dual Hermitian form.

As the angles ϑ_j become larger (but still satisfy the hyperbolicity constraint $(2g-2+n)\pi > \vartheta_1 + \cdots + \vartheta_n$), the situation "deteriorates". In particular, if some $\vartheta_k \geq \pi$, no collar lemma for the conical points holds (see Lemma 6.1). Moreover, for some choice of the hyperbolic metric g on S, there can be no smooth geodesic $\hat{\gamma} \subset S \setminus x$ isotopic to a given loop γ in $S \setminus x$.

As noticed in [**ST08**], $g_{WP,\vartheta}$ becomes smaller as ϑ increases. Moreover, as ϑ_k approaches 2π from below, the fibers of the forgetful map $f_k : \mathcal{T}(S,x) \longrightarrow \mathcal{T}(S, x \setminus \{x_k\})$ (metrically) shrink and $h_{WP,\vartheta}$ converges to $f_k^*(h_{WP,\vartheta_{\hat{k}}})$, where $\vartheta_{\hat{k}} = (\vartheta_1, \dots, \hat{\vartheta}_k, \dots, \vartheta_n)$.

So, for $\vartheta \in [0, 2\pi)^n$ the pairing $h_{WP,\vartheta}$ defines a Kähler metric [**ST05**], but it gets more and more degenerate whenever some ϑ_k overcomes the "walls" $2\pi\mathbb{N}_+$.

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From a different point of view, there is another interesting way to define an alternate pairing on $\mathcal{T}(S,x)$. In fact, a choice of ϑ (such that no ϑ_j is a positive multiple of 2π) permits to real-analytically identify $\mathcal{T}(S,x)$ to the space of Poincaré projective structures (defined by requiring the developing map to be a local isometry) inside the space of all "moderately singular" projective structures $\mathcal{P}(S,x)$ (see Section 4). Moreover, an important theorem of Luo [**Luo93**] (which we reprove in a different way) asserts that, if $\vartheta_k \notin 2\pi\mathbb{N}_+$ for all $1 \le k \le n$, then the holonomy map $\mathcal{P}(S,x) \longrightarrow \mathcal{R}(\pi_1(S\setminus x),\mathrm{PSL}_2(\mathbb{C})) = \mathrm{Hom}(\pi_1(S\setminus x),\mathrm{PSL}_2(\mathbb{C}))/\mathrm{PSL}_2(\mathbb{C})$ is a real-analytic local diffeomorphism.

Our first results, described more extensively in Theorem 4.4, Proposition 4.5 and Proposition 4.6, can be summarized in the following.

THEOREM 1.1. *Let* $\Lambda_- := \{\vartheta \in \mathbb{R}^n_{\ge 0} \,|\, \vartheta_1 + \cdots + \vartheta_n < 2\pi(2g-2+n)\}$ *and* $\Lambda^\circ_- := \Lambda_- \cap (\mathbb{R}_{\ge 0}\setminus 2\pi\mathbb{N}_+)^n$. *Then:*

(a) *the holonomy map* $\mathcal{T}(S,x)\times\Lambda^\circ_- \cong \mathcal{Y}(S,x)(\Lambda^\circ_-) \longrightarrow \mathcal{R}(\pi_1(S\setminus x),\mathrm{PSL}_2(\mathbb{R}))$ *is a real-analytic local diffeomorphism;*
(b) *the restriction of the holonomy map to* $\{\vartheta \in \Lambda_- \,|\, \vartheta_j \le \pi \ \forall j\}$ *is injective;*
(c) *if* $\vartheta_i, \vartheta_j > \pi$ *(for* $i \neq j$*), then the holonomy map* $\mathcal{T}(S,x) \cong \mathcal{Y}(S,x)(\vartheta) \longrightarrow \mathcal{R}(\pi_1(S\setminus x),\mathrm{PSL}_2(\mathbb{R}))$ *is not injective.*

The local behavior around g of the holonomy map can be studied using special coordinates (the a-lengths), namely the hyperbolic lengths of a maximal system of arcs $\boldsymbol{\alpha}$ (which are simple, non-homotopic, non-intersecting unoriented paths between pairs of points in x) adapted to g (see Section 7). Actually, if the angles are smaller than π, the a-lengths allow to reconstruct the full geometry of the surface, so that we can obtain also the injectivity. The existence of adapted triangulations is not obvious if the angles are not small and it is a consequence of the Voronoi decomposition of (S,x) (see Section 8). We remark that, as $\vartheta \to 0$, the Voronoi decomposition and the associated (reduced) a-lengths extend to the space of decorated hyperbolic surfaces with cusps (see Section 6), thus recovering Penner's lambda lengths [**Pen87**].

Back to the previous alternate pairings, the representation space $\mathcal{R}(\pi_1(S\setminus x),\mathrm{PSL}_2(\mathbb{R}))$ is naturally endowed with a Poisson structure η at its smooth points induced by the Lefschetz duality on (S,x) and a $\mathrm{PSL}_2(\mathbb{R})$-invariant nondegenerate symmetric bilinear product on $\mathfrak{sl}_2(\mathbb{R})$ (see Section 5).

Thus, we can compare $\eta_{WP,\vartheta}$ with the pull-back of η via the holonomy map, whenever the angles do not belong to $2\pi\mathbb{N}$. Adapting the work of Goldman [**Gol84**], we prove that the Shimura isomorphism holds for angles smaller than 2π.

THEOREM 1.2. *If* $\vartheta \in \Lambda_- \cap (0,2\pi)^n$, *then*

$$\eta_{WP,\vartheta} = \frac{1}{8}\eta\Big|_{\vartheta}$$

as dual symplectic forms on $\mathcal{Y}(S,x)(\vartheta) \cong \mathcal{T}(S,x)$.

Clearly, we could not ask the equality to hold for larger angles $\vartheta \in \Lambda^\circ_-$, as $\eta_{WP,\vartheta}$ becomes degenerate, whilst $\eta\Big|_{\vartheta}$ is not. However, in proving the theorem we obtain the following.

COROLLARY 1.3. *If* $\vartheta \in \Lambda_-$, *then*

$$\eta_{WP,\vartheta}(\varphi,\psi) = \frac{1}{8}\eta\Big|_{\vartheta}(\varphi,\psi)$$

for $\varphi, \psi \in T^*\mathcal{T}(S,x)$ whenever both hand-sides converge (the right-hand side is always finite if $\vartheta_j \notin 2\pi\mathbb{N}_+$ for all j).

Finally, in Section 9 we find an explicit formula for η in terms of the a-length coordinates.

THEOREM 1.4. *Let $\boldsymbol{\alpha}$ be a triangulation of (S,x) adapted to $g \in \mathcal{Y}(S,x)(\Lambda^\circ_-)$ and let $a_k = \ell_{\alpha_k}$. Then the Poisson structure η at g can be expressed in term of the a-lengths as follows*

$$\eta_g = \sum_{h=1}^{n} \sum_{\substack{s(\overrightarrow{\alpha_i})=x_h \\ s(\overrightarrow{\alpha_j})=x_h}} \frac{\sin(\vartheta_h/2 - d(\overrightarrow{\alpha_i},\overrightarrow{\alpha_j}))}{\sin(\vartheta_h/2)} \frac{\partial}{\partial a_i} \wedge \frac{\partial}{\partial a_j}$$

where $s(\overrightarrow{\alpha}_k)$ is the starting point of the oriented arc $\overrightarrow{\alpha}_k$ and $d(\overrightarrow{\alpha_i},\overrightarrow{\alpha_j})$ is the angle spanned by rotating the tangent vector to the oriented geodesic $\overrightarrow{\hat{\alpha}_i}$ at its starting point clockwise to the tangent vector at the starting point of $\overrightarrow{\hat{\alpha}_j}$.

The techniques are borrowed from Goldman [**Gol86**] and they could be adapted to treat surfaces with boundary or surfaces with conical points and boundary. In fact, the formula is manifestly the analytic continuation of its cousin in [**Mon06**], obtained using techniques of Wolpert [**Wol83**] and the doubling construction (unavalaible here).

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2. Surfaces with constant nonpositive curvature

DEFINITION 2.1. A **pointed surface** (S,x) is a compact oriented surface S of genus g with a nonempty collection $x = (x_1,\dots,x_n)$ of n distinct points on S. We will also write $\dot{S}$ for the punctured surface $S \setminus x$.

We will always assume that $n \geq 3$ if $g = 0$.

Call $\Lambda(S,x)$ the space of (S,x)**-admissible angle parameters**, made of n-tuples $\vartheta = (\vartheta_1,\dots,\vartheta_n) \in \mathbb{R}^n_{\geq 0}$ such that

$$\chi(\dot{S},\vartheta) := (2-2g-n) + \sum_j \frac{\vartheta_j}{2\pi}$$

is nonpositive and we let $\Lambda_-(S,x)$ (resp. $\Lambda_0(S,x)$) be the subset of admissible **hyperbolic** (resp. **flat**) angle parameters, namely those satisfying $\chi(\dot{S},\vartheta) < 0$ (resp. $\chi(\dot{S},\vartheta) = 0$).

We define $\Lambda^\circ(S,x) = \Lambda(S,x) \cap (\mathbb{R} \setminus 2\pi\mathbb{N})^n$ and similarly $\Lambda^\circ_0 := \Lambda_0 \cap \Lambda^\circ$ and $\Lambda^\circ_- = \Lambda_- \cap \Lambda^\circ$. Finally, $\Lambda_{sm}(S,x) := \Lambda(S,x) \cap [0,\pi)^n$ is the subset of **small** angle data.

DEFINITION 2.2. An ϑ**-admissible metric** g on (S,x) is a Riemannian metric of constant curvature on $\dot{S}$ such that, locally around x_j,

$$g = \begin{cases} f(z_j)|z_j|^{2r_j-2}|dz_j|^2 & \text{if } r_j > 0 \text{ or } \chi(\dot{S},\vartheta) = 0 \\ f(z_j)|z_j|^{-2}\log^2|1/z_j|^2|dz_j|^2 & \text{if } r_j = 0 \text{ and } \chi(\dot{S},\vartheta) < 0 \end{cases}$$

where $r_j = \vartheta_j/2\pi$, z_j is a local conformal coordinate at x_j and f is a smooth positive function. A metric g is **admissible** if it is ϑ-admissible for some ϑ.

Remark 2.3. Notice that, if $\chi(\dot{S},\vartheta) < 0$ (or $\vartheta \in \mathbb{R}^n_+$), then such admissible metrics have finite area.

Existence and uniqueness of metrics of nonpositive constant curvature was proven by McOwen [**McO88**] [**McO93**] and Troyanov [**Tro86**] [**Tro91**].

Theorem 2.4 (McOwen, Troyanov). *Given (S,x) and an admissible ϑ as above, there exists a metric of constant curvature on S and assigned angles ϑ at x in each conformal class. Such metric is unique up to rescaling.*

Moreover, Schumacher-Trapani [**ST08**] showed that, for a fixed conformal structure on S, the restriction to a compact subset $K \subset \dot{S}$ of the hyperbolic metric depends smoothly on the associated admissible angle data, provided $\vartheta \in (0,2\pi)^n$.

3. Spaces of admissible metrics

Given a pointed surface (S,x), consider the space of all Riemannian metrics on $\dot{S}$, which is naturally an open convex subset of a Fréchet space. Let $\mathfrak{AMet}(S,x) \subset \mathfrak{Met}(S,x)$ be its subspaces of admissible metrics and of metrics with conical singularities at x. We will deliberately be sloppy about the regularity of such metrics.

The group $\mathrm{Diff}_+(S,x)$ of orientation-preserving diffeomorphisms of S that fix x pointwise clearly acts on $\mathfrak{Met}(S,x)$ preserving $\mathfrak{AMet}(S,x)$.

Definition 3.1. The **Yamabe space** $\widehat{\mathcal{Y}}(S,x)$ is the quotient $\mathfrak{AMet}(S,x)/\mathrm{Diff}_0(S,x)$, where $\mathrm{Diff}_0(S,x) \subset \mathrm{Diff}_+(S,x)$ is the subgroup of isotopies relative to x. Moreover, $\mathcal{Y}(S,x) := \widehat{\mathcal{Y}}(S,x)/\mathbb{R}_+$, where $\mathbb{R}_+$ acts by rescaling.

Remark 3.2. The definition above is clearly modelled on that of Teichmüller space $\mathcal{T}(S,x)$, which is obtained as a quotient of the space of conformal structures $\mathfrak{Conf}(S,x)$ on S by $\mathrm{Diff}_0(S,x)$.

The **mapping class group** $\mathrm{Mod}(S,x) := \mathrm{Diff}_+(S,x)/\mathrm{Diff}_0(S,x)$ acts on $\widehat{\mathcal{Y}}(S,x)$, on $\mathcal{Y}(S,x)$ and on $\mathcal{T}(S,x)$.

There are two natural forgetful maps. The former $\mathfrak{F} : \mathfrak{AMet}(S,x) \longrightarrow \mathfrak{Conf}(S,x)$ only remembers the conformal structure and the latter $\Theta' : \mathfrak{AMet}(S,x) \longrightarrow \Lambda(S,x)$ remembers the angles at the conical points x. They induce $F : \mathcal{Y}(S,x) \longrightarrow \mathcal{T}(S,x)$ and $\Theta : \mathcal{Y}(S,x) \longrightarrow \Lambda(S,x)$ respectively. If $A \subset \Lambda(S,x)$, then we will denote $\Theta^{-1}(A) \subset \mathcal{Y}(S,x)$ by $\mathcal{Y}(S,x)(A)$ for brevity.

Remark 3.3. The forgetful map $(\widetilde{\mathfrak{F}},\widetilde{\Theta}') : \mathfrak{Met}(S,x) \longrightarrow \mathfrak{Conf}(S,x) \times \Lambda(S,x)$ can be given the structure of a fibration in Fréchet or Banach spaces (see for instance [**ST05**]).

Theorem 2.4 says that the restriction of $(\widetilde{\mathfrak{F}},\widetilde{\Theta}')$ to $\mathfrak{AMet}(S,x)$ is a homeomorphism and so its inverse is a section. The following result (due to Schumacher-Trapani) investigates the regularity of this section and uses techniques of implicit function theorem.

Theorem 3.4 ([**ST05**]). *The homeomorphism $(\mathfrak{F},\Theta') : \mathfrak{AMet}(S,x) \longrightarrow \mathfrak{Conf}(S,x) \times \Lambda(S,x)$ restricts to a principal $\mathbb{R}_+$-fibration over $\mathfrak{Conf}(S,x) \times (\Lambda_-(S,x) \cap (0,2\pi)^n)$, and so does $\widehat{\mathcal{Y}}(S,x) \longrightarrow \mathcal{T}(S,x) \times \Lambda(S,x)$. Hence, $(F,\Theta) : \mathcal{Y}(S,x) \longrightarrow \mathcal{T}(S,x) \times \Lambda(S,x)$ restricts to a $\mathrm{Mod}(S,x)$-equivariant homeomorphism over $\mathcal{T}(S,x) \times (\Lambda_-(S,x) \cap (0,2\pi)^n)$.*

A deeper inspection of their proof might show that $(\mathfrak{F}, \Theta')$ restricts to an $\mathbb{R}_+$-fibration over $\mathcal{T}(S,x) \times \Lambda^\circ(S,x)$. In this case, if $\mathcal{Y}(S,x)(\Lambda^\circ(S,x))$ is given the smooth structure coming from Theorem 4.4(a), then (F, Θ) would restrict to a $\mathrm{Mod}(S,x)$-equivariant diffeomorphism over $\mathcal{T}(S,x) \times \Lambda^\circ(S,x)$.

4. Projective structures and holonomy

Let $h_\kappa = \kappa|dw|^2 + |dz|^2$ be a Hermitian product on $\mathbb{C}^2$, with $\kappa \leq 0$, and call $\mathrm{PU}_\kappa \subset \mathrm{PSL}_2(\mathbb{C})$ the projective unitary group associated to h_κ.

Given a pointed surface (S,x), we denote by $\widetilde{\dot{S}} \to \dot{S}$ its universal cover and by $\mathrm{P}T\dot{S} \to \dot{S}$ and $\mathrm{P}T\widetilde{\dot{S}} \to \widetilde{\dot{S}}$ the bundles of real oriented tangent directions. If $\dot{S}$ is endowed with a Riemannian metric, then $\mathrm{P}T\dot{S}$ identifies to the unit tangent bundle $T^1\dot{S}$.

Given an admissible metric g on (S,x) with angles ϑ and curvature κ, one can construct a **developing map** so that the following diagram

$$\begin{tikzcd}
\mathrm{P}T\widetilde{\dot{S}} \arrow[r] \arrow[d] & \mathrm{PU}_\kappa \arrow[r, hook] \arrow[d] & \mathrm{PGL}_2(\mathbb{C}) \arrow[d] \\
\widetilde{\dot{S}} \arrow[r, "\mathrm{dev}"] & D\backslash \mathrm{PU}_\kappa \arrow[r, hook] \arrow[d, "\cong"] & B\backslash \mathrm{PGL}_2(\mathbb{C}) \arrow[d, "\cong"] \\
 & \{v = w/z \in \mathbb{C} \,|\, |v| < 1/\sqrt{|\kappa|}\} \arrow[r, hook] & \mathbb{CP}^1
\end{tikzcd}$$

commutes, where $B \subset \mathrm{PGL}_2(\mathbb{C})$ is the subset of upper triangular matrices and $D = B \cap \mathrm{PU}_\kappa$. In fact, the sphere $S_\kappa := \{(w,z) \in \mathbb{C}^2 \,|\, \kappa|w|^2 + |z|^2 = 1\}$ is acted on by U_κ transitively and its projectivization $\Omega_k := \mathbb{P}S_\kappa$ is still acted on by PU_κ. Hence, $\Omega_\kappa = D\backslash\mathrm{PU}_\kappa$ comes endowed with a metric of curvature κ, so that dev becomes a local isometry.

Remark 4.1. The group PU_κ preserves h_κ and clearly all its nonzero (real) multiples. For $\kappa < 0$, the couple $(\Omega_\kappa, \mathrm{PU}_\kappa)$ is isomorphic to $(\Omega_{-1}, \mathrm{PU}_{-1})$ and so to $(\mathbb{H}, \mathrm{PSL}_2(\mathbb{R}))$. But $D\backslash\mathrm{PU}_0 = \{|z| = 1\}$ and $\kappa^{-1}h_\kappa \to |dw|^2$ as $\kappa \to 0$. Hence, $\Omega_0 \cong \{[w:z] \in \mathbb{CP}^1 \,|\, z \neq 0\} \cong \mathbb{C}$ with the Euclidean metric and

$$\mathrm{PU}_0 \cong \left\{ \begin{pmatrix} u & 0 \\ t & 1 \end{pmatrix} \,\middle|\, u \in \mathrm{U}(1),\ t \in \mathbb{C} \right\} = \{v \mapsto uv + t \,|\, u \in \mathrm{U}(1),\ t \in \mathbb{C}\}$$

We conclude that $(\Omega_0, \mathrm{PU}_0)$ is isomorphic to $(\mathbb{R}^2, \mathrm{SE}_2(\mathbb{R}))$, where $\mathrm{SE}_2(\mathbb{R})$ is the group of affine isometries of $\mathbb{R}^2$ that preserve the orientation.

Let $\mathcal{P}(S,x)$ be the space of **moderately singular projective structures** on $\dot{S}$ (up to isotopy), that is of those whose Schwarzian derivative with respect to the Poincaré structure corresponding to $\vartheta = 0$ has at worst double poles at x. The fibration $p : \mathcal{P}(S,x) \longrightarrow \mathcal{T}(S,x)$ that only remembers the complex structure on S is naturally a principal bundle under the vector bundle $\mathcal{Q}(S,2x) \longrightarrow \mathcal{T}(S,x)$ of holomorphic quadratic differentials (with respect to a conformal structure on S) with at worst double poles at x.

We also call $\mathcal{P}_{con}(S,x)$ the space of **projective structures with conical points**, which are defined to be those moderately singular projective structures that satisfy the following condition: for every j there exists a local holomorphic

coordinate around x_j such that, around $x_j = \{z_j = 0\}$, the universal covering map $\widetilde{\dot{S}} \cong \mathbb{H}_{w_j} \to \dot{S}$ can be written as $w_j \mapsto \exp(iw_j) = z_j$ and the developing map is conjugated to $w_j \mapsto \exp(ir_j w_j)$ for $r_j > 0$ (or to $w_j \mapsto w_j$, if $r_j = 0$). Projective structures with conical points, that admit a developing map whose image is contained in Ω_κ and whose monodromy is a subgroup of PU_κ, are called **admissible** and form a subspace $\mathcal{P}_{adm}(S, x)$.

LEMMA 4.2. *Projective structures with conical points are moderately singular and the Schwarzian derivative between projective structures with the same angle data have zero quadratic residue.*
Hence, every hyperbolic metric with conical points induces an admissible projective structure. Moreover,

$$\begin{tikzcd} \widehat{\mathcal{Y}}(S,x) \arrow[rr, "\widehat{\mathcal{D}}"] \arrow[dr] & & \mathcal{P}(S,x) \\ & \mathcal{Y}(S,x) \arrow[ur, "\mathcal{D}"'] & \end{tikzcd}$$

commutes, $\mathcal{D}$ is a homeomorphism onto $\mathcal{P}_{adm}(S, x)$, which is a closed real-analytic subvariety. Finally, the restriction of $\mathcal{D}$ to each slice $\mathcal{D}_\vartheta : \mathcal{Y}(S, x)(\vartheta) \longrightarrow \mathcal{P}(S, x)$ is a homeomorphism onto a real-analytic subvariety of $\mathcal{P}_{adm}(S, x)$.

PROOF. Admissibility is a simple computation: it turns out that the Schwarzian derivative (with respect to the Poincaré structure with cusps at x) can be written as

$$\boldsymbol{S} = \left[-\frac{1}{2} \left(\frac{\vartheta_j}{2\pi} \right)^2 + O(z_j) \right] \frac{dz_j^2}{z_j^2}$$

where z_j is a local holomorphic coordinate around x_j. Notice also that the Schwarzian derivative of a projective structure with conical singularities ϑ with respect to another projective structure with conical singularities $\widetilde{\vartheta}$ looks like

$$\boldsymbol{S} = \left[\frac{1}{2} \left(\frac{\widetilde{\vartheta}_j^2 - \vartheta_j^2}{(2\pi\vartheta_j)^2} \right) + O(z_j) \right] \frac{dz_j^2}{z_j^2}$$

around x_j (the expression is valid also for $\widetilde{\vartheta}_j = 0$ and $\vartheta_j > 0$). This proves the claim on the residue of $\boldsymbol{S}$.

As the metric can be obtained up to scale by pulling back the metric of Ω_κ via Dev, it follows that $\mathcal{D}$ is bijective. It is easy to check that $\mathcal{D}$ and $\mathcal{D}^{-1}$ are continuous.

Finally, observe that admissible projective structures are characterized by the fact that the image of dev sits in Ω_κ and it has conical singularities at x. The former is a real-analytic closed condition, that can be locally rephrased in terms of holonomy in PU_κ. The latter is also a closed real-analytic condition that can be phrased in terms of quadratic residues of Schwarzian derivative (with respect to the Poincaré structure with cusps at x). A similar argument holds for the image of $\mathcal{D}_\vartheta$. □

REMARK 4.3. It can be proven that $\mathcal{P}_{adm}(S, x)$ is smooth and that the natural map $\mathfrak{AMet}(S, x) \longrightarrow \mathcal{P}_{adm}(S, x)$ is smooth and submersive, which authorizes to

put on $\mathcal{Y}(S,x)$ the smooth structure induced by $\mathcal{P}_{adm}(S,x)$. Thus, $\widehat{\mathcal{Y}}(S,x)$ has a smooth structure too.

Clearly, chosen a base point in $\dot{S}$, we also have an associated **holonomy representation**

$$\rho : \Gamma := \pi_1(\dot{S}) \longrightarrow \mathrm{PU}_\kappa$$

whose image is discrete, for instance, if each $\vartheta_j = 2\pi r_j$ with $1/r_j \in \mathbb{N}_+$. However, for almost all angles ϑ the representation ρ does not have discrete image.

Given a Lie group G, call $\mathcal{R}(\Gamma, G)$ the space $\mathrm{Hom}(\Gamma, G)/G$ of representations up to conjugation.

We will denote by Hol the holonomy map $\mathrm{Hol} : \mathcal{P}(S,x) \longrightarrow \mathcal{R}(\Gamma, \mathrm{PGL}_2(\mathbb{C}))$ (and by abuse of notation, its compositions $\widehat{\mathcal{Y}}(S,x) \to \mathcal{Y}(S,x) \longrightarrow \mathcal{R}(\Gamma, \mathrm{PGL}_2(\mathbb{C}))$ with $\mathcal{D}$) and by hol its "restricted" versions $\mathrm{hol} : \mathcal{Y}(S,x)(\Lambda_-) \longrightarrow \mathcal{R}(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ and $\mathrm{hol} : \mathcal{Y}(S,x)(\Lambda_0) \longrightarrow \mathcal{R}(\Gamma, \mathrm{SE}_2(\mathbb{R}))$, obtained using the isomorphisms $\mathrm{PU}_\kappa \cong \mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{PU}_0 \cong \mathrm{SE}_2(\mathbb{R})$.

Notice that the traces of the holonomies of the boundary loops do not detect the angles $\vartheta \in \mathbb{R}^n$ at the conical points (with the exception of the cusps), but just their class in $(\mathbb{R}/2\pi\mathbb{Z})^n$. Thus, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{Y}(S,x) \arrow[r, "\mathcal{D}"] & \mathcal{P}_{con}(S,x) \arrow[r, "\Theta"] \arrow[d, "\mathrm{Hol}"] & \mathbb{R}^n_{\geq 0} \arrow[d] \\
& \mathcal{R}(\Gamma, \mathrm{PGL}_2(\mathbb{C})) \arrow[r, "\overline{\Theta}"] & (\mathbb{R}/2\pi\mathbb{Z})^n
\end{tikzcd}$$

THEOREM 4.4. *The holonomy maps satisfy the following properties:*

(a) *the restriction* $\mathrm{hol} : \mathcal{P}_{adm}(S,x) \to \mathcal{R}(\Gamma, \mathrm{PGL}_2(\mathbb{C}))$ *to* $\Theta^{-1}(\Lambda^\circ)$ *is a real-analytic immersion and so* $\Theta^{-1}(\Lambda^\circ)$ *is smooth;*

(b) $\mathrm{hol}\Big|_{\Lambda_{sm,-}}$ *and* $\mathrm{hol}\Big|_{\Lambda_{sm,0}}$ *are injective onto open subsets of the corresponding representation spaces.*

Hence, $\mathrm{hol}\Big|_{\Lambda_{sm,-}}$ *and* $\mathrm{hol}\Big|_{\Lambda_{sm,0}}$ *are diffeomorphisms onto their images.*

As a consequence, $\mathrm{Hol}\Big|_{\Lambda_{sm,-}}$ and $\mathrm{Hol}\Big|_{\Lambda_{sm,0}}$ are diffeomorphisms onto their images too.

PROOF. Part (a) was established by Luo [**Luo93**] in greater generality. In the flat case, it was already known to Veech [**Vee93**]. Proposition 7.8 gives a proof for the hyperbolic and flat case that uses lengths of arcs dual to the spine.

Part (b) is a consequence of Lemma 7.3, which guarantees that there exists a (unique) smooth geodesic in each homotopy class of simple closed curves, if the angles are smaller than π, and that its length can be computed from the holonomy representation. Thus, the injectivity follows from the standard reconstruction principle for hyperbolic surfaces which are decomposed into a union of pair of pants. □

Actually, a more careful look shows that, in negative curvature, if $\vartheta_j \leq \pi$ for every j, then pair of pants decompositions still exist, the reconstruction principle works and the holonomy map is still injective. Of course, one must allow "degenerate pair of pants" consisting of one segment, which are obtained by cutting along a simple closed geodesic which separates a couple $\{x_i, x_j\}$ with $\vartheta_i = \vartheta_j = \pi$ from

the rest of the surface and which consists of twice a geodesic segment that joins x_i and x_j.

Even though we will not formalize this approach here, it is intuitive that the failure of the injectivity for $\mathrm{hol}\Big|_{\vartheta}$ is related to the lack of properness of $\mathrm{hol}\Big|_{\vartheta}$ and so to the possibility of extending the holonomy map to some points in the boundary of the augmented Teichmüller space $\overline{\mathcal{T}}(S,x)$ in such a way that the holonomy of a pinched loop is sent to an elliptic element of $\mathrm{PSL}_2(\mathbb{R})$.

In fact, if $J \in \overline{\mathcal{T}}(S,x)$, then $\mathrm{hol}\Big|_{\vartheta}$ for hyperbolic metrics continuously extends to J if and only if we can associated to J a ϑ-admissible metric g in which the only type of degeneration is given by conical points $x_{i_1},\dots,x_{i_k}$ with $\vartheta_{i_1}+\dots+\vartheta_{i_k} > 2\pi(k-1)$ coalescing together. When this singularity occurs, the loop surrounding the coalescing points has elliptic holonomy.

Hence, if there are $i_1,\dots,i_k$ such that $\vartheta_{i_1}+\dots+\vartheta_{i_k} > 2\pi(k-1)$, then *the holonomy map* $\mathrm{hol}\Big|_{\vartheta}$ *is not proper*, but it will become so if we extend it to those points of $\overline{\mathcal{T}}(S,x)$ corresponding to the degenerations mentioned before.

In the flat case, the situation is different as we don't have a collar lemma (see Lemma 6.1), so that injectivity may fail for arbitary small angles. However, as in the hyperbolic case, we do not have properness of the holonomy map if $\vartheta_i+\vartheta_j > 2\pi$ for certain $i \neq j$ (or if $\vartheta_1 > 2\pi$ and $n = 1$).

As an example of the non-injectivity phenomenon we have the following.

Proposition 4.5. (a) *Let $\vartheta \in \Lambda_-$ be angle data such that $\vartheta_h+\vartheta_j > 2\pi$ for certain $h \neq j$. Then* $\mathrm{hol}\Big|_{\vartheta}$ *is not injective.*
(b) *Let $\vartheta \in \Lambda_0$ be angle data such that $\vartheta_h+\vartheta_j \in (2\pi,\infty)\cap\mathbb{Q}$ for certain $h \neq j$. Then* $\mathrm{hol}\Big|_{\vartheta}$ *is not injective.*

Proof. The case in which some angles are positive multiples of 2π are treated in Proposition 4.6, so that we can assume that no holonomy along the loop γ_k that winds around x_k is the identity for all $k = 1,\dots,n$.

Let's analyze case (a). Because $\vartheta_h+\vartheta_j > 2\pi$, there are metrics in which x_h and x_j are at distance $d > 0$ arbitrarily small. Given a metric g, we can assume up to conjugation that

$$\mathrm{hol}(g)(\gamma_h) = \begin{pmatrix} \cos(\tilde{\vartheta}_h/2) & -\sin(\tilde{\vartheta}_h/2) \\ \sin(\tilde{\vartheta}_h/2) & \cos(\tilde{\vartheta}_h/2) \end{pmatrix}$$

$$\mathrm{hol}(g)(\gamma_j) = \begin{pmatrix} \cos(\tilde{\vartheta}_j/2) & -e^{d}\sin(\tilde{\vartheta}_j/2) \\ e^{-d}\sin(\tilde{\vartheta}_j/2) & \cos(\tilde{\vartheta}_j/2) \end{pmatrix}$$

where $\tilde{\vartheta}_j, \tilde{\vartheta}_h \in (0,2\pi)$, $\vartheta_j \equiv \tilde{\vartheta}_j$ and $\vartheta_h \equiv \tilde{\vartheta}_h \mod 2\pi$.

Thus, the loop $\beta := \gamma_j * \gamma_h$ has holonomy $\mathrm{hol}(g)(\beta) = \mathrm{hol}(g)(\gamma_h)\mathrm{hol}(g)(\gamma_j)$ with

$$|\mathrm{Tr}(\mathrm{hol}(g)(\beta))| = 2|\cos(\tilde{\vartheta}_h/2)\cos(\tilde{\vartheta}_j/2) - \cosh(d)\sin(\tilde{\vartheta}_h/2)\sin(\tilde{\vartheta}_j/2)|$$

which is strictly smaller than $2|\cos[(\tilde{\vartheta}_h+\tilde{\vartheta}_j)/2| \leq 2$.

Hence, there exists another metric g' for which such a $d > 0$ is small and $|\mathrm{Tr}(\mathrm{hol}(g')(\beta))| = 2|\cos(\pi p/q)|$, where p,q are positive coprime integers and $p/q < 1$, and so $\mathrm{hol}(g')(\beta)$ has order q.

Let $\tau_\beta \in \mathrm{Mod}(S,x)$ be the Dehn twist along β. If we place the basepoint for π outside the component of $S \setminus \beta$ the contains x_h and x_j, then the action of τ_β on

$\mathcal{R}(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ is trivial on every loop that does not meet β and it is by conjugation by $\mathrm{hol}(\cdot)(\beta)$ on γ_h and γ_j. Hence, τ_β^q fixes $\mathrm{hol}(g')$ but it acts freely on $\mathcal{T}(S,x)$, which shows that the holonomy map is not injective.

The proof of (b) follows the same lines, but it's actually easier. In fact, $\mathrm{hol}(g)(\beta)$ is actually a rotation of angle exactly $\vartheta_1+\vartheta_2-2\pi$ (centered somewhere in the plane). Thus, it is of order q. Hence, τ_β^q acts trivially on $\mathcal{R}(\Gamma, \mathrm{SE}_2(\mathbb{R}))$ but freely on $\mathcal{T}(S,x)$ and the conclusion follows. □

A suitable modification of part (a) of the above proof would also show that injectivity would similarly fail if $n = 1$ and $\vartheta_1 > 2\pi$.

Our feeling is that the non-injectivity of the holonomy map in negative curvature is only associated to the phenomenon above. It would be interesting to make this precise.

Another interesting issue is to understand when the images of $\mathrm{hol}\Big|_{\vartheta}$ and $\mathrm{hol}\Big|_{\widetilde{\vartheta}}$ intersect. For instance, if all angles are integral multiples of 2π, then the holonomy representation descends to $\mathcal{R}(\pi_1(S), \mathrm{PSL}_2(\mathbb{R}))$ and Milnor-Wood's inequality allows us to recover $\vartheta_1 + \cdots + \vartheta_n$. Given a ϑ-admissible hyperbolic metric g, the question then becomes whether $\mathrm{hol}(g)$ remembers at least the area of g.

The last piece of information about the holonomy maps concerns what happens when some angles are *integral*, i.e. integral multiples of 2π, and so the corresponding holonomies are the identity.

Proposition 4.6. *Let $G = \mathrm{PSL}_2(\mathbb{R})$ (if $\chi(\dot{S}, \vartheta) < 0$) or $G = \mathrm{SE}_2(\mathbb{R})$ (if $\chi(\dot{S}, \vartheta) = 0$).*

1. *If $\vartheta \in \Lambda^\circ(S,x)$, then $\mathrm{hol}\Big|_{\vartheta} : \mathcal{Y}(S,x)(\vartheta) \longrightarrow \mathcal{R}(\Gamma, G)$ is a locally closed real-analytic diffeomorphism onto its image.*
2. *If $\vartheta_j = 2\pi$, then $\mathrm{hol}\Big|_{\vartheta} : \mathcal{Y}(S,x)(\vartheta) \cong \mathcal{T}(S,x) \longrightarrow \mathcal{R}(\Gamma, G)$ is constant along the fibers of the forgetful map $\mathcal{T}(S,x) \to \mathcal{T}(S, x \setminus \{x_j\})$.*
3. *If $\vartheta_j = 2\pi r_j$ with $r_j \geq 1$ integer and if z_j is a holomorphic coordinate on S around x_j such that locally $\mathrm{dev}(z_j) = z_j^{r_j} + b$, then the differential of $\mathrm{hol}\Big|_{\vartheta} : \mathcal{P}_{con}(S,x)(\vartheta) \longrightarrow \mathcal{R}(\Gamma, G)$ vanishes along the tangent directions determined by deforming the local developing map around x_j as $\mathrm{dev}_\varepsilon(z_j) = b + (z_j + \varepsilon c z_j^{1-r_j})^{r_j} + o(\varepsilon) = b + z_j^{r_j} + r_j c\varepsilon + o(\varepsilon)$, for every $c \in \mathbb{C}$. Hence, the differential of $\mathrm{hol}\Big|_{\vartheta} : \mathcal{Y}(S,x)(\vartheta) \cong \mathcal{T}(S,x) \longrightarrow \mathcal{R}(\Gamma, G)$ vanishes along the first-order Schiffer variation $c z_j^{1-r_j} \frac{\partial}{\partial z_j}$.*

We recall that a Schiffer variation of complex structure on (S, J) is defined as follows. Let $D_j \subset S$ be a disc centered at x_j and let z_j be a holomorphic coordinate on D_j so that $z_j(D_j) = \{z \in \mathbb{C} \,|\, |z| < 1\}$; call $D_{i,\delta} := \{p \in D_j \,|\, |z_j(p)| < \delta\}$. Given a holomorphic vector field $V = f(z_j)\partial/\partial z_j$ on $\dot{D}_j$ with a pole in x_j, we can define a new Riemann surface $(S_\varepsilon, J_\varepsilon)$ (which is canonically diffeomorphic to S up to isotopy) by gluing D_j and $(S \setminus D_{j,1/2}) \cup g_\varepsilon(D_j)$ through the map $g_\varepsilon : D_j \setminus D_{j,\delta} \to S \setminus \{x_j\}$ given by $z \mapsto z + \varepsilon f(z)$, which is a biholomorphism onto its image for ε small enough.

A simple argument shows that the tangent direction in $T_J\mathcal{T}(S,x) \cong H^{0,1}_J(S, T_S(-x))$ determined by such a Schiffer variation does not depend on the disc D_j and on δ,

but only on the jet of V at x_j. In particular, we have

$$0 \longrightarrow H^0(S, T_S(-x+\infty x_j)) \longrightarrow \hat{\mathcal{M}}_{S,x_j}/\hat{\mathcal{O}}_{S,x_j}(T_S(-x)) \longrightarrow H^{0,1}(S, T_S(-x)) \longrightarrow 0$$

where $\hat{\mathcal{O}}_{S,x_j}$ is the completed local ring of functions at x_j and $\hat{\mathcal{M}}_{S,x_j}$ is its field of fractions. More naively, elements in $\hat{\mathcal{M}}_{S,x_j}/\hat{\mathcal{O}}_{S,x_j}(T_S(-x))$ can be represented as $(\sum_{-m\leq k\leq 0} c_k z_j^k)\partial/\partial z_j$.

Proof of Proposition 4.6. Part (1) is clearly a consequence of Theorem 4.4(a).

For part (3), notice that the holonomy around x_j is trivial. Thus, the vector field $cz_j^{1-r_j}\frac{\partial}{\partial z_j}$ that deforms the local developing map as $z_j \mapsto (z_j + \varepsilon c z_j^{1-r_j})^{r_j} = z_j^{r_j} + r_j c\varepsilon + o(\varepsilon)$ produces a deformation of projective structure which fixes the holonomy. Clearly, (2) follows from (3). □

Remark 4.7. Notice that a simultaneous Schiffer variation at $x_1, \dots, x_n$ with vector fields $V_1, \dots, V_n$ determine the zero tangent vector only if they extend to a global section of T_S (holomorphic on $S \setminus x$), and this can happen only if $m_1 + \dots + m_n \geq 2g-2+n$, where $m_j = \mathrm{ord}_{x_j}(V_j)$. Thus, if $\chi(\dot{S}, \vartheta) < 0$ or if $\vartheta_j \notin 2\pi\mathbb{N}_+$ for some j, then any first-order deformation of an admissible metric that fixes holonomy changes the conformal structure.

5. Poisson structures

Now, we will implicitly represent each class in $\mathcal{Y}(S,x)(\Lambda_-)$ by a metric g of curvature -1, so that the (restricted) holonomy map gives a representation $\rho : \Gamma = \pi_1(\dot{S}) \longrightarrow \mathrm{PSL}_2(\mathbb{R})$. Because of the choice of a base-point, ρ is only well-defined up to conjugation by $\mathrm{PSL}_2(\mathbb{R})$.

On the other hand, we also have a local system $\xi \longrightarrow \dot{S}$ defined by $\xi = (\widetilde{\dot{S}} \times \mathfrak{g})/\Gamma$, where $\widetilde{\dot{S}}$ is the universal cover of $\dot{S}$, $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ is the Lie algebra of $\mathrm{PSL}_2(\mathbb{R})$ and Γ acts on $\widetilde{\dot{S}}$ via deck transformations and on $\mathfrak{g}$ via ρ and the adjoint representation. Let $D_1, \dots, D_n \subset S$ be open disjoint discs such that $x_j \in D_j$ and call $D = \bigcup_j D_j$. We will slightly abuse notation by denoting still by ξ the restriction of $\xi \to \dot{S}$ to $\dot{D}$.

We recall that $\mathfrak{B}(X,Y) := \mathrm{Tr}(XY)$ for $X, Y \in \mathfrak{g}$ is a nondegenerate symmetric bilinear form of signature $(2,1)$. Given

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

then $\{H, E+F, E-F\}$ is a $\mathfrak{B}$-orthogonal basis of $\mathfrak{g}$, with $\mathfrak{B}(H,H) = \mathfrak{B}(E+F, E+F) = 2$ and $\mathfrak{B}(E-F, E-F) = -2$. Notice that $E-F$ generates the rotations around $i \in \mathbb{H}$. Actually, $\mathfrak{K} = -4\mathfrak{B}$, where $\mathfrak{K}$ is the Killing form on $\mathfrak{g}$. Denote still by $\mathfrak{B}$ the induced pairing on $\mathfrak{g}^*$.

Deforming the (conjugacy class of the) representation ρ is equivalent to deforming the (isomorphism class of the) local system ξ.

As shown for instance in [**Gol84**], first-order deformations of $\rho \in \mathcal{R}(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ are parametrized by $H^1(\dot{S}; \xi)$. Thus, $T_\rho\mathcal{R}(\Gamma, \mathrm{PSL}_2(\mathbb{R})) \cong H^1(\dot{S}; \xi)$ and dually $T^*_\rho\mathcal{R}(\Gamma, \mathrm{PSL}_2(\mathbb{R})) \cong H_1(\dot{S}; \xi^*)$, which is isomorphic to $H^1(\dot{S}, \dot{D}; \xi)$ by Lefschetz duality (and the nondegeneracy of $\mathfrak{B}$).

When no $\vartheta_j \in 2\pi\mathbb{N}_+$, the long exact sequence in cohomology for the couple $(\dot{S}, \dot{D})$ give rise to the following identifications

$$\begin{tikzcd}
0 \arrow[r] & H^0(\dot{D};\xi) \arrow[r] \arrow[d, "\wr", no head, equal] & H^1(\dot{S},\dot{D};\xi) \arrow[r] \arrow[d, "\wr", no head, equal] & H^1(\dot{S};\xi) \arrow[r] \arrow[d, "\wr", no head, equal] & H^1(\dot{D};\xi) \arrow[r] \arrow[d, "\wr", no head, equal] & 0 \\
0 \arrow[r] & (\mathbb{R}^n)^* \arrow[r, "(d\overline{\Theta})^*"] & T^*_\rho\mathcal{R}(\Gamma,\mathrm{PSL}_2(\mathbb{R})) \arrow[r, "\eta"] & T_\rho\mathcal{R}(\Gamma,\mathrm{PSL}_2(\mathbb{R})) \arrow[r, "d\overline{\Theta}"] & \mathbb{R}^n \arrow[r] & 0
\end{tikzcd}$$

where $g \in \mathcal{Y}(S,x)(\Lambda_-^\circ)$ and $H^0(\dot{S};\xi) \cong H^2(\dot{S},\dot{D};\xi)^* = 0$ because ρ has no fixed vectors.

Notice that, if g is a ϑ-admissible metric, the parabolic cohomology group $H^1_P(\dot{S};\xi)$ at $\rho = \mathrm{hol}(g)$, defined as the image of $H^1(\dot{S},\dot{D};\xi) \to H^1(\dot{S};\xi)$ identifies (via hol) to the space of those first-order deformations of metrics (equivalently, of projective structures) with conical singularities along which ϑ is constant.

At a point ρ such that $\vartheta_j \in 2\pi\mathbb{N}_+$, we have $H^0(\dot{D}_j;\xi) \cong \mathfrak{g} \cong H^1(\dot{D}_j;\xi)$ and so $\mathcal{R}(\Gamma,\mathrm{PSL}_2(\mathbb{R}))$ is singular at such a ρ. In this case, there are deformations of ρ which correspond to opening a hole or creating a cusp at x_j. Conversely, if no $\vartheta_j \in 2\pi\mathbb{N}_+$, then $\mathrm{hol}(g)$ lies in the smooth locus of $\mathcal{R}(\Gamma,\mathrm{PSL}_2(\mathbb{R}))$.

Though not completely trivial, the following result can be obtained adapting arguments from [**AB83**], [**Gol84**] or [**Kar92**], who proved that η defines a symplectic structure if x is empty.

Lemma 5.1. *The alternate pairing η defines a Poisson structure on the smooth locus of $\mathcal{R}(\Gamma,\mathrm{PSL}_2(\mathbb{R}))$. Hence, the pull-back of η through* hol *defines a Poisson structure on $\mathcal{Y}(S,x)(\Lambda_-^\circ) \cong \mathcal{T}(S,x) \times \Lambda_-^\circ(S,x)$, which will still be denoted by η.*

The second part follows from the fact that hol is a local diffeomorphism (Theorem 4.4(a)).

As already investigated by Goldman [**Gol84**] in the case of closed surfaces, it is natural to explore the relation between η and the **Weil-Petersson pairing**, which is defined as $\eta_{WP,\vartheta} := \mathrm{Im}(h^*_{WP})$, where

$$h^*_{WP,\vartheta}(\varphi,\psi) := -\frac{1}{4}\int_S g_\vartheta^{-1}(\varphi,\overline{\psi})$$

g_ϑ^{-1} is the dual hyperbolic Kähler form on S with angle data ϑ and $\varphi,\psi \in H^0(S, K_S^{\otimes 2}(x))$ are cotangent vectors to $\mathcal{T}(S,x) \cong \mathcal{Y}(S,x)(\vartheta)$ at g.

For angles smaller than 2π, the Shimura isomorphism still holds.

Theorem 5.2. *If $\vartheta \in \Lambda_-(S,x) \cap (0,2\pi)^n$, then*

$$\eta_{WP,\vartheta} = -\frac{1}{8}\eta\Big|_\vartheta$$

as dual symplectic forms on $\mathcal{Y}(S,x)(\vartheta) \cong \mathcal{T}(S,x)$.

Schumacher-Trapani [**ST08**] have also shown that, if $\vartheta \in (0,2\pi)^n$, then $\eta^*_{WP,\vartheta}$ is a Kähler form and that $\eta^*_{WP,\vartheta}$ degenerates in the expected way as some $\vartheta_j \to 2\pi$.

Proof of Theorem 5.2. Mimicking [**Gol84**], we consider the diagram

$$\begin{tikzcd}
 & \xi = \mathrm{dev}^*\mathfrak{g} \arrow[d, "\mathrm{dev}^*\sigma"] \\
T_{\dot{S}} \arrow[r, "\beta"] & \mathrm{dev}^*T_{\mathbb{H}}
\end{tikzcd}$$

in which $\sigma : \mathfrak{g} \to T_{\mathbb{H}}$ maps $\mathfrak{g}$ to the $\mathrm{SL}_2(\mathbb{R})$-invariant vector fields of $\mathbb{H}$.

If $r_j = \vartheta_j/2\pi > 0$, then dev locally looks like

$$\mathrm{dev}: \; z_j \mapsto i\,\frac{1-z_j^{r_j}}{1+z_j^{r_j}}$$

up to action of $\mathrm{PSL}_2(\mathbb{R})$ for some holomorphic local coordinate z_j around x_j. So

$$\beta := d(\mathrm{dev}) = -\frac{2ir_j z_j^{r_j-1}}{(1+z_j^{r_j})^2}$$

Moreover, if w is the standard coordinate on $\mathbb{H} = \{w = s + it \,|\, s,t \in \mathbb{R},\ t > 0\}$, then

$$\mathfrak{B}\sigma = \begin{pmatrix} w & -w^2 \\ 1 & -w \end{pmatrix} \frac{\partial}{\partial w}$$

thus, around $z_j = 0$ we have that

$$\tau := \beta^{-1} \circ \mathrm{dev}^*(\mathfrak{B}\sigma) = \Big[i(1 - \frac{2z_j^{r_j}}{1+z_j^{r_j}})H + (2 - \frac{4z_j^{r_j}}{(1+z_j^{r_j})^2})(E+F) + \\ - \frac{4z_j^{r_j}}{(1+z_j^{r_j})^2}(E-F) \Big] \frac{i(1+z_j^{r_j})^2}{2r_j z_j^{r_j-1}} \frac{\partial}{\partial z_j}$$

belongs to $H^0(S, T_S(\sum_j (r_j-1)x_j) \otimes \xi)$. Moreover, the dual Kähler form associated to the Poincaré metric on $\mathbb{H}$

$$g_{\mathbb{H}}^{-1} = t^2 \frac{\partial}{\partial s} \wedge \frac{\partial}{\partial t} = -2it^2 \frac{\partial}{\partial w} \wedge \frac{\partial}{\partial \overline{w}}$$

can be recovered as $g_{\mathbb{H}}^{-1} = (i/2)\mathrm{Tr}(\mathfrak{B}\sigma \wedge \mathfrak{B}\overline{\sigma})$, where

$$\mathfrak{B}\sigma \wedge \mathfrak{B}\overline{\sigma} = \begin{pmatrix} |w|^2 - w^2 & (w-\overline{w})|w|^2 \\ \overline{w} - w & |w|^2 - \overline{w}^2 \end{pmatrix} \frac{\partial}{\partial w} \wedge \frac{\partial}{\partial \overline{w}}$$

Hence, $g_{\vartheta}^{-1} = -(i/2)\mathfrak{B}(\tau \wedge \overline{\tau})$.

As we can identify $T\mathcal{P}_{adm}(S,x)$ and $T\mathcal{R}(\Gamma, \mathrm{PSL}_2(\mathbb{R})) \cong H^1(\dot{S};\xi)$ via $d$hol, then the restriction of $p : \mathcal{P}(S,x) \to \mathcal{T}(S,x)$ to $\mathcal{P}_{adm}(S,x)$ can be infinitesimally described as follows. Given $\nu \in H^1(\dot{S};\xi)$, we can look at its restrictions $\nu_j \in H^1(\dot{D}_j;\xi)$. If $\nu_j = 0$, then ν does not vary the angle ϑ_j and so there is a representative for ν that vanishes on $\dot{D}_j$. If $\nu_j \neq 0$, then it can be represented by a Cech 1-cocycle with locally costant coefficients in ξ. As the $(E-F)$-component of τ is $-\frac{2iz_j}{r_j}\frac{\partial}{\partial z_j}$, we conclude that $\tau\nu_j$ has a representative that vanishes at x_j. Hence, $\tau\nu$ has always a representative that vanishes at x, whose class in $H^1(S, T_S(-x))$ will be denoted by $\widetilde{\tau\nu}$, and $dp : T\mathcal{R}(\Gamma, \mathrm{PSL}_2(\mathbb{R})) \to \mathcal{T}(S,x)$ incarnates into

$$\begin{array}{ccc} H^1(\dot{S};\xi) & \longrightarrow & H^1(S, T_S(-x)) \\ \nu & \longmapsto & \mathfrak{B}(\widetilde{\tau\nu}) \end{array}$$

which is the restriction to real projective structures of the map $H^1(\dot{S};\xi_{\mathbb{C}}) \longrightarrow H^1(S, T_S(-x))$ still given by $\nu \mapsto \mathfrak{B}(\widetilde{\tau\nu})$. Its dual is thus

$$\begin{array}{ccc} H^0(S, K_S^{\otimes 2}(x)) & \longrightarrow & H^1(\dot{S}, \dot{D}; \xi_{\mathbb{C}}) \\ \varphi & \longmapsto & \widetilde{\varphi\tau} \end{array}$$

where $\widetilde{\varphi\tau}$ can be represented by $\xi_{\mathbb{C}}$-valued 1-form cohomologous to $\varphi\tau$, which vanishes on $\dot{D}$, whose existence depends on the fact that no $\vartheta_j \in 2\pi\mathbb{N}$ and so $\varphi\tau$ has no residue at x. A similar formula holds for real projective structures.

Hence, it is easy now to see that, if all the terms are convergent, then

$$h^*_{WP,\vartheta} = -\frac{1}{4}\int_S g_\vartheta^{-1}(\tau,\overline{\psi}) = \frac{i}{8}\int_S \mathfrak{B}(\varphi\tau\wedge\overline{\psi\tau}) = \frac{i}{8}\int_S \mathfrak{B}(\widetilde{\varphi\tau}\wedge\overline{\widetilde{\psi\tau}}) =$$
$$= \frac{i}{8}[S]\cap\mathfrak{B}(p^*(\varphi)\cup\overline{p^*(\psi)})$$

As we are working with real projective structures, $\overline{\psi\tau} = \psi\tau$ and this concludes the argument. □

Notice that, as $\vartheta_j > 2\pi$ increases, the Weil-Petersson pairing on $T\mathcal{Y}(S,x)(\vartheta)$ becomes more and more degenerate, the walls being given exactly by $\vartheta_j \in 2\pi\mathbb{N}$. However, the above proof also yields the following.

Corollary 5.3. *If $\vartheta \in \Lambda_-(S,x)$ and $\varphi,\psi \in T^*\mathcal{T}(S,x)$, then*

$$\eta_{WP,\vartheta}(\varphi,\psi) = \frac{1}{8}\eta\Big|_\vartheta(\varphi,\psi)$$

whenever both hand-sides are convergent.

6. Decorated hyperbolic surfaces

Let $\vartheta_{max} = \max\{\vartheta_1,\dots,\vartheta_n\}$ and recall the collar lemma for hyperbolic surfaces with conical points.

Lemma 6.1 (Dryden-Parlier [**DP07**]). *If $\vartheta \in \Lambda_{sm,-}(S,x)$, then there exists $R \in (0,1]$ which depends only on $\vartheta_{max} < \pi$ such that, for every hyperbolic metric g on S with angles ϑ at x, the balls B_j centered at x_j with circumference $\le R$ are disjoint and do not meet any simple closed geodesic.*

We call such balls B_j *small*. The following definition is inspired by Penner [**Pen87**], who first introduced decorated hyperbolic surfaces with cusps. Notice that a class in $\mathcal{Y}(S,x)$ will be usually represented by an admissible metric of curvature -1.

Definition 6.2. A **decoration** for a hyperbolic surface (S,x) with small angle data ϑ is the choice of small balls $B_1,\dots,B_n$ (not all reduced to a point); equivalently, of the nonzero vector $\varepsilon = (\varepsilon_1,\dots,\varepsilon_n) \in [0,R)^n$ of their circumferences.

Remark 6.3. Notice that a hyperbolic surface S with small angles ϑ can be given a *standard decoration* by letting B_j to be the ball of radius $s(\vartheta) = \cosh^{-1}(1/\sin(\vartheta_{max}/2))/2$. The constant is chosen in such a way that the area of $B := B_1\cup\dots\cup B_n$ is bounded from below (by a positive constant) for all hyperbolic structures on S (with angle ϑ). The circumference of B_j is clearly $s(\vartheta)\vartheta_j$.

Thus, the assignment of $[s(\vartheta)\vartheta]$ defines a map $\mathcal{Y}(S,x)\setminus\Theta^{-1}(0) \longrightarrow \mathbb{P}(\mathbb{R}^n_{\ge 0})$. The closure of its graph identifies to the real-oriented blow-up $\mathrm{Bl}_0\mathcal{Y}(S,x)$ and the exceptional divisor $\Theta^{-1}(0)\times\mathbb{P}(\mathbb{R}^n_{\ge 0})$ can be understood as the space of hyperbolic metrics with cusps on $\dot{S}$ (up to isotopy) together with a **projective decoration** $[\varepsilon] \in \mathbb{P}(\mathbb{R}^n_{\ge 0})$, which plays the role of infinitesimal angle datum. Clearly, a projective decoration $[\varepsilon]$ is canonically represented by the **normalized decoration** ε in its

class, obtained by prescribing $\varepsilon_1 + \cdots + \varepsilon_n = 1$; so we can identify $\mathbb{P}(\mathbb{R}^n_{\geq 0})$ with Δ^{n-1}.

Thus, the map Θ lifts to $\widehat{\Theta} : \mathrm{Bl}_0\mathcal{Y}(S,x) \longrightarrow \Delta^{n-1} \times [0, 2\pi(2g-2+n)]$. We remark that a similar projective decoration arises in [**Mon06**] as infinitesimal boundary length datum.

7. Arcs

Given a pointed surface (S,x), we call **arc** the image $\alpha = f(I)$ of a continuous $f : (I, \partial I) \to (S,x)$, in which $I = [0,1]$ and f injectively maps $\mathring{I}$ into $\dot{S}$. Let $\mathfrak{Arc}_0(S,x)$ be the space of arcs with the compact-open topology and let $\mathfrak{Arc}_n(S,x)$ be the subset of $\mathfrak{Arc}_0(S,x)^{(n+1)}$ consisting of unordered pairwise non-homotopic (relative to x) $(n+1)$-tuple of arcs $\boldsymbol{\alpha} = \{\alpha_0, \dots, \alpha_n\}$ such that $\alpha_i \cap \alpha_j \subset x$ for $i \neq j$.

Remark 7.1. Equivalently, we could have defined $\mathfrak{Arc}'_0(S,x)$ to be the space of unoriented simple closed free loops γ in $S \setminus x$ which are homotopy equivalent to an arc α (i.e. such that $\gamma = \partial U_\alpha$, where U is a tubular neighbourhood of α). We could have defined $\mathfrak{Arc}'_n(S,x)$ analogously. Clearly, $\mathfrak{Arc}'_n(S,x) \simeq \mathfrak{Arc}_n(S,x)$. We will also say that $\alpha_1, \alpha_2 \in \mathfrak{Arc}_0(S,x)$ are homotopic *as arcs* if they belong to the same connected component.

Notice that each $\mathfrak{Arc}_n(S,x)$ is contractible, because $\chi(\dot{S}) < 0$.

Definition 7.2. A **$(k+1)$-arc system** is an element of $\mathfrak{A}_k(S,x) := \pi_0(\mathfrak{Arc}_k(S,x))$. A **triangulation** is a maximal system of arcs $\boldsymbol{\alpha} \in \mathfrak{A}_{N-1}(S,x)$, where $N = 6g - 6 + 3n$.

Notice that, if $\boldsymbol{\alpha} = \{\alpha_i\}$ is a triangulation, then its **complement** $S \setminus \boldsymbol{\alpha} := S \setminus \bigcup_i \alpha_i$ is a disjoint union of triangles.

Lemma 7.3. *Let α_i be an arc and g be a ϑ-admissible metric on (S,x).*

(1) *There exist a geodesic $\hat{\alpha}_i \subset S$ and a homotopy $\alpha_i(t) : I \to S$ with fixed endpoints such that $\alpha_i(0) = \alpha_i$, $\alpha_i(1) = \hat{\alpha}_i$ and $\mathrm{int}(\alpha_i(t)) \cap x = \emptyset$ for $t \in [0,1)$ and $\mathrm{int}(\alpha_i(1)) \cap x$ can only contain points x_j such that $\vartheta_j \geq \pi$.*
(2) *If two geodesic arcs $\hat{\alpha}_i$ and $\hat{\alpha}'_i$ are homotopic as arcs, then they are equal.*
(3) *If all $\vartheta_j < \pi$, then for each α_i there exists exactly one smooth geodesic $\hat{\alpha}_i$ homotopic to α_i as an arc.*

The second assertion is a consequence of the nonpositivity of the curvature and (3) follows from (1) and (2). To prove (1), one takes a minimizing sequence in the homotopy class of α_i and a limit $\hat{\alpha}_i$ of such a sequence (S is compact). One immediately concludes by looking at the geometry of a conical point. Whether or not the (possibly broken) geodesic $\hat{\alpha}_i$ obtained in (1) is an arc, we will still say by abuse of notation that $\hat{\alpha}_i$ is *the unique geodesic homotopic to* α_i.

Definition 7.4. An arc α_i on (S,x) is **compatible** with the metric g if there exists a smooth geodesic $\hat{\alpha}_i$, which is homotopic to α_i as arcs.

Let $p \in \alpha_i^\circ \subset \dot{S}$ and let $\gamma_b, \gamma_c \in \pi_1(\dot{S}, p)$ be loops that wind around x_b, x_c such that $\gamma_b * \gamma_c$ corresponds to α_i. If $\mathrm{dev} : \tilde{\dot{S}} \to \Omega$ is the developing map (where $\Omega = \mathbb{H}, \mathbb{C}$), then call $\tilde{x}_b, \tilde{x}_c$ the endpoints of $\tilde{\alpha}_i := \mathrm{dev}(\alpha'_i)$, where α'_i is a lift of α_i to $\tilde{\dot{S}}$.

Definition 7.5. The **a-length** associated to an arc α_i is the function $a_i : \widehat{\mathcal{Y}}(S,x) \longrightarrow [0,\infty]$ defined as the distance between $\tilde{x}_b$ and $\tilde{x}_c$.

Remark 7.6. Notice that, if the angles at x_b and x_c are not integral multiples of 2π, then $\tilde{x}_b$ and $\tilde{x}_c$ are the unique fixed points in $\mathbb{H}$ of $\mathrm{Hol}(g)(\gamma_b)$ and $\mathrm{Hol}(g)(\gamma_c)$. Hence, Lemma A.2(a) and Lemma A.3(a) ensure that a_i is real-analytic around g, where $a_i > 0$ (i.e. where $\tilde{x}_b \neq \tilde{x}_c$). Moreover, if α_i is compatible with g, then $a_i(g)$ is the g-length of $\hat{\alpha}_i$. In general, the length $\hat{a}_i$ of the (broken) geodesic $\hat{\alpha}_i$ homotopic to α_i is positive and piecewise real-analytic: in fact, $\hat{\alpha}_i$ is locally equal to the join of finitely many smooth geodesic arcs $\hat{\alpha}_{i_1} * \cdots * \hat{\alpha}_{i_k}$ and so $\hat{a}_i = a_{i_1} + \cdots + a_{i_k}$.

Given a triangulation $\boldsymbol{\alpha}$, the a-lengths associated to the unique hyperbolic metric define a map

$$\ell_{\boldsymbol{\alpha}} : \mathcal{Y}(S,x) \longrightarrow \mathrm{Bl}_0[0,\infty]^N$$

where the infinitesimal a-lengths Δ^{N-1} arise in particular when the surface becomes flat.

If (S,x,B) is a surface with hyperbolic metric g, small ϑ and a normalized decoration B, then we can define the **reduced a-length** of an α_i that joins x_b and x_c to be $\tilde{a}_i := a_i - (\varepsilon_b + \varepsilon_c)$, where $\varepsilon_b, \varepsilon_c$ are the radii of B_b, B_c. If α_i is compatible with g, then $\tilde{a}_i = \ell_{\hat{\alpha}_i \setminus B}$. Because of the standard decoration mentioned in Remark 6.3 for metrics with small angles, the reduced a-lengths can be extended to an open neighbourhood of $\widehat{\Theta}^{-1}(0)$.

Definition 7.7. A triangulation $\boldsymbol{\alpha}$ of (S,x) is **adapted** to the ϑ-admissible metric $g \in \mathrm{Bl}_0\mathcal{Y}(S,x)$ if:

(a) every $\alpha_i \in \boldsymbol{\alpha}$ is compatible with g;
(b) if $\vartheta \neq 0$, then there is only one directed arc in $\boldsymbol{\alpha}$ outgoing from each cusp (resp. from each cylinder, if $\chi(\dot{S},\vartheta) = 0$);
(c) if $\vartheta = 0$ and $[\varepsilon]$ is the projective decoration, then there is only one directed arc in $\boldsymbol{\alpha}$ outgoing from those x_j with $\varepsilon_j = 0$.

We remark that, if $\vartheta \in [0,\pi)^n$, then the compatibility condition (a) is automatically satisfied. The utility of adapted triangulations relies on the following result, which directly follows from the above considerations.

Proposition 7.8. *Let $\boldsymbol{\alpha}$ be triangulation adapted to $g \in \mathcal{Y}(S,x) \setminus \Theta^{-1}(0)$ (resp. $(g,[\varepsilon]) \in \Theta^{-1}(0) \subset \mathrm{Bl}_0\mathcal{Y}(S,x)$) and suppose that $\vartheta_j \notin 2\pi\mathbb{N}_+$, where $\vartheta = \Theta(g)$.*

(1a) *If $0 \neq \vartheta \in \Lambda_-(S,x)$, then $a_i = \ell_{\alpha_i}$ is a real-analytic function of $\mathrm{Hol}(g) \in \mathcal{R}(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ in a neighbourhood of g.*
(1b) *If $0 \neq \vartheta \in \Lambda_0(S,x)$, then $\dfrac{a_i}{a_j} = \dfrac{\ell_{\alpha_i}}{\ell_{\alpha_j}}$ is a real-analytic function of $\mathrm{Hol}(g) \in \mathcal{R}(\Gamma, \mathrm{SE}_2(\mathbb{R}))/\mathbb{R}_+$ in a neighbourhood of $g \in \Theta^{-1}(\Lambda_0)$.*
(2) *If $\vartheta = 0$, then $\tilde{a}_i = \tilde{\ell}_{\alpha_i}$ is a real-analytic function of $\mathrm{Hol}(g) \in \mathcal{R}(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ and $[\varepsilon]$ in a neighbourhood of $(g,[\varepsilon]) \in \Theta^{-1}(0)$.*

Because hyperbolic (resp. Euclidean) triangles are characterized by the lengths of their edges (resp. by the projectivization of the Euclidean lengths of their edges), it is thus clear that the holonomy together with an adapted triangulation allow to reconstruct the full geometry of the surface.

Corollary 7.9. *Let $\boldsymbol{\alpha}$ be a triangulation on (S,x).*

(1a) *If* $\boldsymbol{\alpha}$ *is adapted to* $g \in \mathcal{Y}(S,x)(\Lambda_-) \setminus \Theta^{-1}(0)$, *then* $\ell_{\boldsymbol{\alpha}}$ *is a local system of real-analytic coordinates on* $\mathcal{Y}(S,x)$ *around* g.

(1b) *If* $\boldsymbol{\alpha}$ *is adapted to* $g \in \mathcal{Y}(S,x)(\Lambda_0)$, *then* $\left\{ \frac{\ell_{\alpha_i}}{\ell_{\alpha_1}} \,\middle|\, i = 2, \dots, N \right\} \cup \{\chi(\dot{S}, \vartheta)\}$ *is a local system of coordinates on* $\mathcal{Y}(S,x)$ *around* g.

(2) *If* $\boldsymbol{\alpha}$ *is adapted to* $(g, [\varepsilon]) \in \widehat{\Theta}^{-1}(0)$, *then* $\widetilde{\ell}_{\boldsymbol{\alpha}}$ *is a local system of real-analytic coordinates on* $\mathrm{Bl}_0\mathcal{Y}(S,x)$ *around* $(g, [\varepsilon])$.

The next task will be to produce at least one triangulation adapted to g for every $g \in \mathrm{Bl}_0\mathcal{Y}(S,x)$.

8. Voronoi decomposition

Let (S,x) be a surface with a ϑ-admissible metric g. For the moment, we assume $\Theta(g) \neq 0$, so that the function $dist : \dot{S} \to \mathbb{R}_{\geq 0}$ that measures the distance from x is well-defined.

Definition 8.1. A **shortest path** from $p \in \dot{S}$ is a (geodesic) path from p to x of length $dist(p)$.

The concept of shortest path can extended to the whole S. In fact, it is clear that at every x_j with $\vartheta_j > 0$ the constant path is the only shortest one.

Remark 8.2. If x_j is marks a cusp (resp. a cylinder), then we can cure our definition as follows. Consider a horoball B_j around x_j of small circumference (resp. a semi-infinite cylinder B_j ending at x_j), so that no other conical points sit inside B_j and all simple geodesics that enter B_j end at x_j. Let γ be a nonconstant geodesic from x_j to x, which is made of two portions: γ' from x_j to the first intersection point y of $\gamma \cap B_j$ and $\gamma'' = \gamma \setminus \gamma'$. We say that γ is **shortest** if $\ell_{\gamma''} = dist(y)$. One can easily see that there are finitely many shortest paths from a cusp (resp. a cylinder) and that there is at least one (because ∂B_j is compact).

If ϑ is small, then we can consider the modified distance (with sign) $\widetilde{dist} : S \to [-\infty, \infty]$ of a point in S from ∂B, where B is the standard decoration and $\widetilde{dist}(p)$ is positive if and only if $p \in S \setminus B$. Mimicking the trick as in the previous remark, we can define a modified valence function $\widetilde{\mathrm{val}}$ on the whole S. It is clear that $\mathrm{val} = \widetilde{\mathrm{val}}$.

Thus, we can define $\widetilde{d}$ and $\widetilde{\mathrm{val}}$ on a projectively decorated surface $(S, x, [\varepsilon])$, by choosing a system of small balls B whose projectivized circumferences are $[\varepsilon]$.

Definition 8.3. The **valence** $\mathrm{val}(p)$ of a point $p \in S$ is the number of shortest paths at p. The **Voronoi graph** $G(g)$ is the locus of points of valence at least two.

Because g has constant curvature, one can conclude that $G(g)$ is a finite one-dimensional CW-complex embedded inside $\dot{S}$ with geodesic edges: its vertices are $V(g) = \mathrm{val}^{-1}([3, \infty))$ and its (open) edges are $E(g) = \pi_0(\mathrm{val}^{-1}(2))$. Notice that the closure $\overline{G(g)}$ passes through x_j if and only if $\vartheta_j = 0$.

By definition, for every edge $e \in E(g)$ and for every $p \in e$, there are exactly two shortest paths $\overrightarrow{\beta_1}(p)$ and $\overrightarrow{\beta_2}(p)$ from p. Moreover, the interior of $\overrightarrow{\beta_i}(p)$ does not contain any other marked point for $i = 1, 2$. Then the composition $\alpha_e(p) := \overleftarrow{\beta_1}(p) * \overrightarrow{\beta_2}(p)$ is an arc from some x_j to some x_j and its homotopy class (as arcs) α_e is independent of p.

REMARK 8.4. The angle $\psi_0(e)$ at x_j spanned by $\bigcup_{p\in e}\overleftarrow{\beta_1}(p)$ is called "edge invariant" by Luo [**Luo08**].

DEFINITION 8.5. The (isotopy class of the) path $\alpha_e \subset S$ is the **arc dual to** $e \in E(g)$ and $\boldsymbol{\alpha}(g) = \{\alpha_e \,|\, e \in E\}$ is the **Voronoi system of arcs** for g.

The complement $S \setminus \boldsymbol{\alpha}(g) := \bigcup_{v\in V} t_v$ is called **Voronoi decomposition**. The cell t_v is a pointed polygon if v is a cusp and it is a polygon otherwise.

PROPOSITION 8.6. *Let $g \in \mathrm{Bl}_0\mathcal{Y}(S,x)$ be a hyperbolic/flat admissible metric (resp. a hyperbolic admissible metric with a projective decoration $[\varepsilon]$) and let $\boldsymbol{\alpha}(g)$ its Voronoi system. Consider a maximal system of arcs $\boldsymbol{\alpha} \supseteq \boldsymbol{\alpha}(g)$ such that only one oriented arc in $\boldsymbol{\alpha}$ terminates at each cusp/cylinder (resp. at each cusp x_j with $\varepsilon_j = 0$). Then*

1. *α_i is compatible with g;*
2. *the geodesic representative $\hat{\alpha}_i$ of each $\alpha_i \in \boldsymbol{\alpha}$ intersects x only at $\partial\hat{\alpha}_i$;*
3. *$\boldsymbol{\alpha}$ is adapted to g.*

PROOF. We only deal with the case $\Theta \neq 0$. The decorated case is similar and so we omit the details.

Suppose that $\hat{\alpha}_i$ joins x_j to x_k (possibly $j = k$). Let e be the edge of the Voronoi graph $G(g)$ dual to α_i (which may reduce to a vertex) and call v_0 the point of e which is closest to x_j and x_k. Let $\overrightarrow{\beta_j}(v_0)$ (resp. $\overrightarrow{\beta_k}(v_0)$) be the shortest path from v_0 to x_j (resp. x_k), so that $\alpha_i \simeq \overleftarrow{\beta_j}(v_0) * \overrightarrow{\beta_k}(v_0)$.

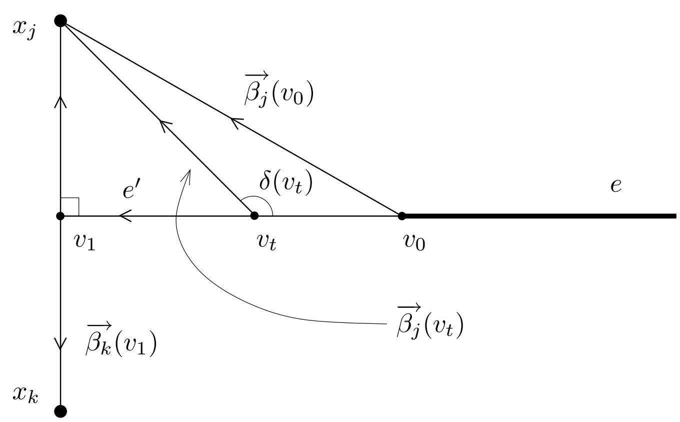

FIGURE 1. The case in which $e' \neq \{v_0\}$.

Consider the maximal closed geodesic segment e' that starts at v_0 and such that, for every $v \in e'$, the shortest path $\overrightarrow{\beta_j}(v)$ from v to x_j homotopic to $\overrightarrow{vv_0} * \overrightarrow{\beta_j}(v_0)$ and the shortest path $\overrightarrow{\beta_k}(v)$ from v to x_k homotopic to $\overrightarrow{vv_0} * \overrightarrow{\beta_k}(v_0)$ satisfy $\ell(\beta_j(v)) = \ell(\beta_k(v)) \leq \ell(\beta_j(v_0)) = \ell(\beta_k(v_0))$. Call $\delta(v)$ the angle $\widehat{v_0 v \beta_j} = \widehat{v_0 v \beta_k}$.

If $e' = \{v_0\} \subset e$, then $\delta(v_0) = \pi/2$ and $\mathrm{int}(\beta_j(v_0)) \cap x = \mathrm{int}(\beta_k(v_0)) \cap x = \emptyset$; so $\overleftarrow{\beta_j}(v_0) * \overrightarrow{\beta_k}(v_0)$ is already the desired smooth geodesic $\hat{\alpha}_i$.

Otherwise, start travelling along e' from v_0 until the point v_1 which is closest to x_j and x_k. Call v_t the points of e' between v_0 and v_1 for $t \in (0,1)$. Clearly, $\delta(v_1) = \pi/2$ and $\delta(v_t)$ is a strictly decreasing function of t.

As a consequence, $d(v_0, y) < d(v_0, x_j)$ for all $y \in \mathrm{int}(\beta_j(v_t))$ and $t \in (0,1]$ (and similarly for x_k). Thus, $\mathrm{int}(\beta_j(v_t)) \cap x = \mathrm{int}(\beta_k(v_t)) \cap x = \emptyset$ for $t \in [0,1]$.

We can conclude that $\alpha_i(t) := \overleftarrow{\beta_j}(v_t) * \overrightarrow{\beta_k}(v_t)$ is the wished homotopy of arcs between $\alpha_i \simeq \alpha_i(0)$ and the smooth geodesic $\hat{\alpha}_i := \alpha_i(1)$.

Parts (2) and (3) clearly follow from (1). □

REMARK 8.7. It was shown by Rivin [**Riv94**] (in the flat case) and by Leibon [**Lei02**] (in the hyperbolic case) that the Voronoi construction gives a $\mathrm{Mod}(S,x)$-equivariant cellularization of $\mathcal{Y}(S,x)$: the affine coordinates on each cell are given by $\{\psi_0(e) \,|\, e \in E(g)\}$ (Luo [**Luo06**] has shown that one can also use different curvature functions ψ_k). This is similar to what happens for surfaces with geodesic boundary, after replacing ψ_0 by the analogous quantity [**Luo07**] [**Mon06**]. However, the cone parameters $\psi_0(e)$ must obey some extra constraints, because the sum of the internal angles of a triangle t cannot exceed π. Thus, the cells of $\mathcal{Y}(S,x)$ are *truncated* simplices.

9. An explicit formula

Similarly to [**Pen92**] and [**Mon06**], we want now to provide an explicit formula for η in terms of the a-lengths, using techniques from [**Gol86**].

THEOREM 9.1. *Let $\boldsymbol{\alpha}$ be a triangulation of (S,x) adapted to $g \in \mathcal{Y}(S,x)(\Lambda^\circ_-)$ and let $a_k = \ell_{\alpha_k}$. Then the Poisson structure η at g can be expressed in terms of the a-lengths as follows*

$$\eta_g = \sum_{h=1}^{n} \sum_{\substack{s(\overrightarrow{\alpha_i})=x_h \\ s(\overrightarrow{\alpha_j})=x_h}} \frac{\sin(\vartheta_h/2 - d(\overrightarrow{\alpha_i}, \overrightarrow{\alpha_j}))}{\sin(\vartheta_h/2)} \frac{\partial}{\partial a_i} \wedge \frac{\partial}{\partial a_j}$$

where $s(\overrightarrow{\alpha}_k)$ is the starting point of the oriented geodesic arc $\overrightarrow{\alpha}_k$ and $d(\overrightarrow{\alpha_i}, \overrightarrow{\alpha_j})$ is the angle spanned by rotating the tangent vector to $\overrightarrow{\alpha_i}$ at its starting point clockwise to the tangent vector at the starting point of $\overrightarrow{\alpha_j}$. If $\vartheta \in (0, 2\pi)^n$, then the above formula also expresses 8-times the Weil-Petersson dual symplectic form $\eta_{WP,\vartheta}$ at $g \in \mathcal{T}(S,x)$.

REMARK 9.2. In [**Mon06**] a similar formula for hyperbolic surfaces with geodesic boundary is proven. Really, if Σ is a surface with boundary, and $d\Sigma$ is its double with the natural real involution σ, then $\pi_\iota : \mathcal{T}(d\Sigma)^\sigma \to \mathcal{T}(\Sigma)$ has the property that $(\pi_\iota)_* \eta_{WP,dS} = 2\eta_{WP,S}$, and not $\eta_{WP,S}$, as claimed in Proposition 1.7 of [**Mon06**]. This explains why the two formulae are off by a factor 2.

PROOF OF THEOREM 9.1. We want to compute $\eta_g(da_i, da_j)$. Fix a basepoint $p \in \dot{S}$ and call $\gamma(\overrightarrow{\alpha_k})$ the parabolic element of $\Gamma := \pi_1(\dot{S}, p)$ that winds around $s(\overrightarrow{\alpha_k})$, in such a way that $\gamma(\overrightarrow{\alpha_k}) * \gamma(\overleftarrow{\alpha_k})$ corresponds to the arc α_k.

Let $\rho := \mathrm{Hol}(g)$ and let $u \in H^1(\dot{S}; \xi)$ be a tangent vector in $T_\rho \mathcal{R}(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$. The deformation of ρ corresponding to u can be written as $\rho_t(\gamma) = \rho(\gamma) + tu(\gamma)\rho(\gamma) + O(t^2)$ and we will also write $S_k(t) = \rho_t(\gamma(\overrightarrow{\alpha_k}))$ and $s_k = \log(S_k)$, and similarly $F_k(t) = \rho_t(\gamma(\overleftarrow{\alpha_k}))$ and $f_k = \log(F_k)$.

Because of Lemma 9.3(c),

$$\mathfrak{B}(da_i, da_j) = \frac{4\mathfrak{B}(d\mathfrak{B}(s_i, f_i) \cap d\mathfrak{B}(s_j, f_j))}{\sinh(a_i)\sinh(a_j)\vartheta_{s(\overrightarrow{\alpha_i})}\vartheta_{s(\overleftarrow{\alpha_i})}\vartheta_{s(\overrightarrow{\alpha_j})}\vartheta_{s(\overleftarrow{\alpha_j})}}$$

The numerator potentially contains 4 summands: we will only compute the one occurring when $s(\overrightarrow{\alpha_i}) = s(\overrightarrow{\alpha_j})$, as the others will be similar. In particular, because of Lemma 9.3(b), we need to calculate $\mathfrak{B}(R_i \otimes \gamma(\overrightarrow{\alpha_i}) \cap R_j \otimes \gamma(\overrightarrow{\alpha_j}))$, where $R_k := (1-\mathrm{Ad}_{S_k}^{-1})^{-1}[f_k, s_k]$, because $s_k \otimes \gamma(\overrightarrow{\alpha_k})$ (resp. $f_k \otimes \gamma(\overleftarrow{\alpha_k})$) is a multiple of $d\vartheta_{s(\overrightarrow{\alpha_k})}$ (resp. $d\vartheta_{s(\overleftarrow{\alpha_k})}$) by Lemma 9.3(a) and $d\vartheta_h$ belongs to the radical of η for every h.

The local situation around $s(\overrightarrow{\alpha_i})$ is described in Figure 2.

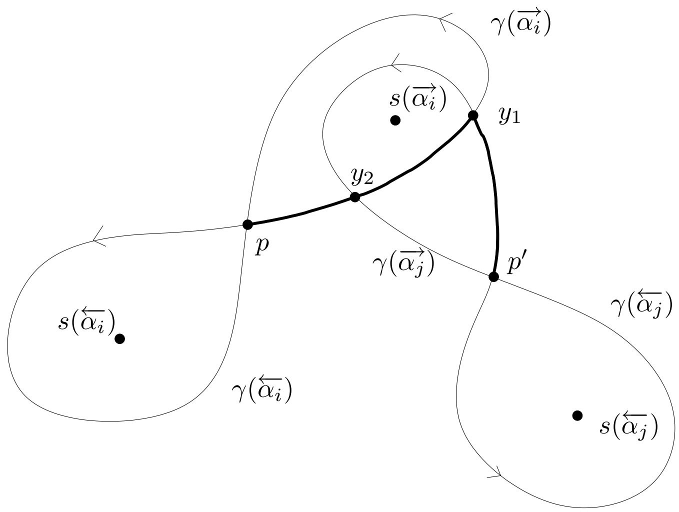

FIGURE 2. The bundle ξ is trivialized along the thick path.

The intersection pairing at the level of 1-chains gives $\gamma(\overrightarrow{\alpha_i}) \cap \gamma(\overrightarrow{\alpha_j}) = y_1 - y_2$. Because we have trivialized ξ on the thick part, we obtain

$$\mathfrak{B}(R_i \otimes \gamma(\overrightarrow{\alpha_i}) \cap R_j \otimes \gamma(\overrightarrow{\alpha_j})) = \mathfrak{B}(R_i, (1-\mathrm{Ad}_{S_j}^{-1})R_j) = \mathfrak{B}(R_i, [f_j, s_j])$$

By Lemma A.2,

$$[s_k, f_k] = \frac{1}{4}\vartheta_{s(\overrightarrow{\alpha_k})}\vartheta_{s(\overleftarrow{\alpha_k})}[L(S_k), L(F_k)] = \frac{1}{2}\vartheta_{s(\overrightarrow{\alpha_k})}\vartheta_{s(\overleftarrow{\alpha_k})}\sinh(a_k)L(\overrightarrow{\alpha_k})$$

where $L(\overrightarrow{\alpha_k})$ is the axis of the geodesic $\overrightarrow{\hat{\alpha}_k}$.

So far we have obtained

$$\begin{aligned}\mathfrak{B}((1-\mathrm{Ad}_{S_i^{-1}})^{-1}[f_i, s_i], [f_j, s_j]) = \frac{1}{4}\vartheta_{s(\overrightarrow{\alpha_i})}\vartheta_{s(\overleftarrow{\alpha_i})}\vartheta_{s(\overrightarrow{\alpha_j})}\vartheta_{s(\overleftarrow{\alpha_j})}\sinh(a_i)\sinh(a_j)\cdot \\ \cdot\, \mathfrak{B}((1-\mathrm{Ad}_{S_i}^{-1})^{-1}L(\overrightarrow{\alpha_i}), L(\overrightarrow{\alpha_j}))\end{aligned}$$

Notice that $\mathrm{Ad}_{S_i^h} = \exp(h\,\mathrm{ad}_{s_i})$ acts on $L(\overrightarrow{\alpha_i})$ as a rotation of angle $h\nu$ centered at $s(\overrightarrow{\alpha_i})$, where $\nu = \vartheta_{s(\overrightarrow{\alpha_i})}$, and so

$$\mathfrak{B}(\mathrm{Ad}_{S_i^h}L(\overrightarrow{\alpha_i}), L(\overrightarrow{\alpha_j})) = 2\cos(-\delta + h\nu) = 2\mathrm{Re}\left[\exp((-\delta)\sqrt{-1} + h\nu\sqrt{-1})\right]$$

where $\delta = d(\overrightarrow{\alpha_i}, \overrightarrow{\alpha_j})$. Hence,

$$\mathfrak{B}(w(\mathrm{ad}_{s_i})L(\overrightarrow{\alpha_i}), L(\overrightarrow{\alpha_j})) = 2\mathrm{Re}\left[\exp(-\delta\sqrt{-1})w(\nu\sqrt{-1})\right]$$

where w is an analytic function.

Therefore, we can conclude that

$$\mathfrak{B}(R_i \otimes \gamma(\overrightarrow{\alpha_i}) \cap R_j \otimes \gamma(\overrightarrow{\alpha_j})) = \frac{1}{4} \vartheta_{s(\overrightarrow{\alpha_i})} \vartheta_{s(\overleftarrow{\alpha_i})} \vartheta_{s(\overrightarrow{\alpha_j})} \vartheta_{s(\overleftarrow{\alpha_j})} \sinh(a_i) \sinh(a_j) \frac{\sin(\vartheta_{s(\overrightarrow{\alpha_i})}/2 - \delta)}{\sin(\vartheta_{s(\overrightarrow{\alpha_i})}/2)}$$

because $2\mathrm{Re}\left[\dfrac{\exp(-\delta\sqrt{-1})}{1-\exp(-\nu\sqrt{-1})}\right] = \dfrac{\sin(\nu/2-\delta)}{\sin(\nu/2)}$.

Finally, the first summand of $\mathfrak{B}(da_i, da_j)$ is $\dfrac{\sin(\vartheta_{s(\overrightarrow{\alpha_i})}/2 - d(\overrightarrow{\alpha_i}, \overrightarrow{\alpha_j}))}{\sin(\vartheta_{s(\overrightarrow{\alpha_i})}/2)}$. □

To complete the proof of the theorem, we only need to establish the following.

Lemma 9.3.

(a) $d\vartheta_{s(\overrightarrow{\alpha_k})} = L(S_k) \otimes \gamma(\overrightarrow{\alpha_k})$

(b) $d\mathfrak{B}(s_k(t), f_k(t)) = (1 - \mathrm{Ad}_{F_k^{-1}})^{-1}[s_k, f_k] \otimes \gamma(\overleftarrow{\alpha_k}) + (1 - \mathrm{Ad}_{S_k^{-1}})^{-1}[f_k, s_k] \otimes \gamma(\overrightarrow{\alpha_k}) + $
$+ \frac{\mathfrak{B}(f_k,f_k)}{\mathfrak{B}(s_k,f_k)} f_k \otimes \gamma(\overleftarrow{\alpha_k}) + \frac{\mathfrak{B}(s_k,s_k)}{\mathfrak{B}(f_k,s_k)} s_k \otimes \gamma(\overrightarrow{\alpha_k})$

(c) $\sinh(a_k) da_k = \left[\dfrac{2d\vartheta_{s(\overrightarrow{\alpha_k})}}{\vartheta^2_{s(\overrightarrow{\alpha_k})}\vartheta_{s(\overleftarrow{\alpha_k})}} + \dfrac{2d\vartheta_{s(\overleftarrow{\alpha_k})}}{\vartheta^2_{s(\overleftarrow{\alpha_k})}\vartheta_{s(\overrightarrow{\alpha_k})}}\right] \mathfrak{B}(s_k, f_k) - \dfrac{2d\mathfrak{B}(s_k, f_k)}{\vartheta_{s(\overrightarrow{\alpha_k})}\vartheta_{s(\overleftarrow{\alpha_k})}}$

as elements of $T^*_g \mathcal{Y}(S, x) \cong H_1(\dot{S}; \xi)$.

Proof. Part (a) was essentially proved in [**Gol86**] and part (c) is easily obtained from Lemma A.2(a) by differentiation.

For part (b), consider the function $\mathfrak{B}(s_k(t), f_k(t))$ along the path $t \mapsto \rho_t = \exp(tu)\rho = \rho + tu\rho + O(t^2)$, where $s_k(0) = s_k$ and $f_k(0) = f_k$. By Lemma A.4

$$s_k(t) = \log\left[\exp(tu_{\overrightarrow{k}}) \exp(s_k)\right] = s_k + t(1 - \mathrm{Ad}_{S_k})^{-1}[s_k, u_{\overrightarrow{k}}] + t\frac{\mathfrak{B}(u_{\overrightarrow{k}}, s_k)}{\mathfrak{B}(s_k, s_k)} + O(t^2)$$

where $u_{\overrightarrow{k}} = u(\gamma(\overrightarrow{\alpha_k}))$ and $u_{\overleftarrow{k}} = u(\gamma(\overleftarrow{\alpha_k}))$. Hence,

$$\begin{aligned}
\mathfrak{B}(s_k(t), f_k(t)) = \mathfrak{B}(s_k, f_k) &+ t\mathfrak{B}(s_k, (1 - \mathrm{Ad}_{F_k})^{-1}[f_k, u_{\overleftarrow{k}}]) + t\frac{\mathfrak{B}(u_{\overleftarrow{k}}, f_k)}{\mathfrak{B}(f_k, f_k)}\mathfrak{B}(s_k, f_k) + \\
&+ t\mathfrak{B}(f_k, (1 - \mathrm{Ad}_{S_k})^{-1}[s_k, u_{\overrightarrow{k}}]) + t\frac{\mathfrak{B}(u_{\overrightarrow{k}}, s_k)}{\mathfrak{B}(s_k, s_k)}\mathfrak{B}(f_k, s_k) + O(t^2) = \\
= \mathfrak{B}(s_k, f_k) &+ t\mathfrak{B}(u_{\overleftarrow{k}}, (1 - \mathrm{Ad}_{F_k^{-1}})^{-1}[s_k, f_k]) + t\frac{\mathfrak{B}(f_k, f_k)}{\mathfrak{B}(s_k, f_k)}\mathfrak{B}(u_{\overleftarrow{k}}, f_k) + \\
&+ t\mathfrak{B}(u_{\overrightarrow{k}}, (1 - \mathrm{Ad}_{S_k^{-1}})^{-1}[f_k, s_k]) + t\frac{\mathfrak{B}(s_k, s_k)}{\mathfrak{B}(f_k, s_k)}\mathfrak{B}(u_{\overrightarrow{k}}, s_k) + O(t^2)
\end{aligned}$$

Finally,

$$\begin{aligned}
d\mathfrak{B}(s_k(t), f_k(t)) = (1 - \mathrm{Ad}_{F_k^{-1}})^{-1}[s_k, f_k] \otimes \gamma(\overleftarrow{\alpha_k}) + (1 - \mathrm{Ad}_{S_k^{-1}})^{-1}[f_k, s_k] \otimes \gamma(\overrightarrow{\alpha_k}) + \\
+ \frac{\mathfrak{B}(f_k, f_k)}{\mathfrak{B}(s_k, f_k)} f_k \otimes \gamma(\overleftarrow{\alpha_k}) + \frac{\mathfrak{B}(s_k, s_k)}{\mathfrak{B}(f_k, s_k)} s_k \otimes \gamma(\overrightarrow{\alpha_k})
\end{aligned}$$

□

Appendix A. Some linear algebra

Let $R \in \mathrm{PSL}_2(\mathbb{R})$ be a hyperbolic element corresponding to the oriented geodesic $\overrightarrow{\beta}$ in $\mathbb{H}$. Define $L(R) = 2r/\ell(R) \in \mathfrak{sl}_2(\mathbb{R})$, where $r = \log(R)$ is the unique logarithm of R in $\mathfrak{sl}_2(\mathbb{R})$ and $\ell(R) = \operatorname{arccosh}(\mathrm{Tr}(R^2)/2)$ is the translation distance of R, so that $\mathfrak{B}(L(R), L(R)) = 2$.

Remark A.1. Given an oriented hyperbolic geodesic $\overrightarrow{\beta}$ in $\mathbb{H}$, we say that a component of $\mathbb{H} \setminus \beta$ is the *β-positive half-plane* if it induces the orientation of $\overrightarrow{\beta}$ on its boundary. The definition of positive half-plane with respect to an oriented line in $\mathbb{R}^2$ is similar.

If $S \in \mathrm{PSL}_2(\mathbb{R})$ is elliptic of angle $\nu = \arccos(\mathrm{Tr}(S^2)/2)$, then define $L(S) = 2s/\nu \in \mathfrak{sl}_2(\mathbb{R})$, where $s = \log(S)$ is an infinitesimal counterclockwise rotation, so that $\mathfrak{B}(L(S), L(S)) = -2$.

Simple considerations of hyperbolic geometry give the following (see [**Rat06**], for instance).

Lemma A.2. (a) *Let $S_1, S_2 \in \mathrm{PSL}_2(\mathbb{R})$ be elliptic elements that fix distinct points $x_1, x_2 \in \mathbb{H}$ and let R be the hyperbolic element that fixes the unique geodesic through x_1 and x_2 and takes x_1 to x_2. Then*

$$\mathfrak{B}(L(S_1), L(S_2)) = -2\cosh(d(x_1, x_2))$$
$$[L(S_1), L(S_2)] = 2\sinh(d(x_1, x_2))L(R)$$

where $d(x_1, x_2)$ is the hyperbolic distance between x_1 and x_2.

(b) *Let $R_1, R_2 \in \mathrm{PSL}_2(\mathbb{R})$ be hyperbolic elements corresponding to oriented geodesics $\overrightarrow{\beta_1}, \overrightarrow{\beta_2}$ on $\mathbb{H}$. Then*

$$\mathfrak{B}(L(R_1), L(R_2)) = \begin{cases} 2\cos(\delta) & \text{if they meet forming an angle } \delta \\ 2\cosh(d(\beta_1, \beta_2)) & \text{if they are disjoint.} \end{cases}$$

(c) *Let $R \in \mathrm{PSL}_2(\mathbb{R})$ be a hyperbolic element corresponding to $\overrightarrow{\beta}$ and $S \in \mathrm{PSL}_2(\mathbb{R})$ be an elliptic element that fixes $x \in \mathbb{H}$. Then*

$$\mathfrak{B}(L(R), L(S)) = -2\sinh(d(\overrightarrow{\beta}, x))$$

where $d(\overrightarrow{\beta}, x)$ is positive if x lies in the $\overrightarrow{\beta}$-positive half-plane.

In the flat case, we will only need the following simple result.

Lemma A.3. (a) *Let $S_1, S_2 \in \mathrm{SE}_2(\mathbb{R})$ be elliptic elements, namely $S_i(v) = N_i(v) + w_i$ with $1 \neq N_i \in \mathrm{SO}_2(\mathbb{R})$ and $w_i \in \mathbb{R}^2$ for $i = 1, 2$. Thus, S_i has a fixed point $x_i = (1 - N_i)^{-1}w_i$ and the Euclidean distance $d(x_1, x_2)$ can be expressed as*

$$d(x_1, x_2) = \left\|(1 - N_1)^{-1}w_1 - (1 - N_2)^{-1}w_2\right\|$$

(b) *Given elliptic elements $S_1, S_2, S_3 \in \mathrm{SE}_2(\mathbb{R})$ with fixed points x_1, x_2, x_3, then the quantity*

$$x_1 \wedge x_2 + x_2 \wedge x_3 + x_3 \wedge x_1 \in \Lambda^2\mathbb{R}^2 \cong \mathbb{R}$$

is positive (resp. negative, or zero) if and only if x_3 lies in the positive half-plane with respect to the line determined by $\overrightarrow{x_1 x_2}$ (resp. the negative half-plane, or the three points are collinear).

Finally, the following explicit expression is needed in the proof of Lemma 9.3.

Lemma A.4. *Let $s, u \in \mathfrak{sl}_2(\mathbb{R})$ such that s is elliptic or hyperbolic and let $S = \exp(s)$. Then*

$$\log(\exp(tu)S) = s + t(1-\mathrm{Ad}_S)^{-1}[u,s] + t\frac{\mathfrak{B}(u,s)}{\mathfrak{B}(s,s)}s + O(t^2)$$

where $(1-\mathrm{Ad}_S)$ is here interpreted as an automorphism of $s^\perp \subset \mathfrak{sl}_2(\mathbb{R})$.

Proof. Extend $\mathfrak{B}$ to $\mathfrak{gl}_2(\mathbb{R})$, so that $\mathfrak{B}(x,y) = \mathrm{Tr}(xy)$ for $x, y \in \mathfrak{gl}_2(\mathbb{R})$, and consider $(1-\mathrm{Ad}_S) \in \mathrm{End}(\mathfrak{gl}_2(\mathbb{R}))$.

Because s is elliptic or hyperbolic, then $s^2 = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$ with $c \neq 0$, and so $\mathfrak{B}(s,s) \neq 0$. Hence, $V := \ker(1-\mathrm{Ad}_S) = \mathrm{span}\{1, s\}$ and $\mathfrak{gl}_2(\mathbb{R}) = V \oplus W$ is an orthogonal decomposition, where $W = \mathrm{Im}(1-\mathrm{Ad}_S)$.

Notice also that multiplying by s (and so by S or S^{-1}) on the left or on the right is an automorphism of $\mathfrak{gl}_2(\mathbb{R})$ that preserves V and W. Define $M_S : \mathfrak{gl}_2(\mathbb{R}) \longrightarrow \mathfrak{gl}_2(\mathbb{R})$ as

$$M_S(x+y) := (1-\mathrm{Ad}_S)\Big|_W^{-1}(x) \qquad \text{where } x \in W \text{ and } y \in V$$

Clearly, the multiplication by s (or by S or S^{-1}) commutes with Ad_S, and so also with M_S.

Because the first-order term in t in the equality we want to prove is also linear in u, it is sufficient to compute the exponential E of the right hand side (up to $O(t^2)$) in two different cases: $u = s$ and $u \in W$, since $\mathfrak{sl}_2(\mathbb{R}) = W \oplus \mathbb{R}s$.

For $u = s$, we have $[u,s] = 0$ and so

$$\begin{aligned} E &= \exp\left(s + t\frac{\mathfrak{B}(s,s)}{\mathfrak{B}(s,s)}s\right) = \exp(s+ts) = \\ &= S\exp(ts) = S(1+ts+O(t^2)) = S + tsS + O(t^2) \end{aligned}$$

If $u \in W$, then $(1-\mathrm{Ad}_S)(u), (1-\mathrm{Ad}_S)(uS) \in W$. Hence,

$$\begin{aligned} E &= \exp(s + t(1-\mathrm{Ad}_S)^{-1}[u,s]) = \\ &= S + t\sum_{h\geq 1}\frac{1}{h!}\sum_{j=0}^{h-1} s^j M_S^{-1}([u,s])s^{h-1-j} + O(t^2) = \\ &= S + t\sum_{h\geq 1}\frac{1}{h!}\sum_{j=0}^{h-1} M_S^{-1}(s^j[u,s]s^{h-1-j}) + O(t^2) = \\ &= S + t\sum_{h\geq 1} M_S^{-1}([u, s^h/h!]) + O(t^2) = \\ &= S + tM_S^{-1}(uS - Su) + O(t^2) = S + tM_S^{-1}(1-\mathrm{Ad}_S)(uS) + O(t^2) = \\ &= S + tuS + O(t^2). \end{aligned}$$

□

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